

Scattering and bound states of a spin-1/2 neutral particle in the cosmic string spacetime

Fabiano M. Andrade,^{1,2,*} Cleverson Filgueiras,^{3,†} and Edilberto O. Silva^{4,‡}

¹*Department of Computer Science and Department of Physics and Astronomy,
University College London, WC1E 6BT London, United Kingdom*

²*Departamento de Matemática e Estatística, Universidade Estadual de Ponta Grossa, 84030-900 Ponta Grossa-PR, Brazil*

³*Departamento de Física, Universidade Federal de Lavras,
Caixa Postal 3037, 37200-000, Lavras-MG, Brazil*

⁴*Departamento de Física, Universidade Federal do Maranhão, 65085-580 São Luís-MA, Brazil*

(Dated: February 1, 2017)

In this paper the relativistic quantum dynamics of a spin-1/2 neutral particle with a magnetic moment μ in the cosmic string spacetime is reexamined by applying the von Neumann theory of self-adjoint extensions. Contrary to previous studies where the interaction between the spin and the line of charge is neglected, here we consider its effects. This interaction gives rise to a point interaction: $\nabla \cdot \mathbf{E} = (2\lambda/\alpha)\delta(r)/r$. Due to the presence of the Dirac delta function, by applying an appropriated boundary condition provided by the theory of self-adjoint extensions, irregular solutions for the Hamiltonian are allowed. We address the scattering problem obtaining the phase shift, S-matrix and the scattering amplitude. The scattering amplitude obtained shows a dependency with energy which stems from the fact that the helicity is not conserved in this system. Examining the poles of the S-matrix we obtain an expression for the bound states. The presence of bound states for this system has not been discussed before in the literature.

PACS numbers: 03.65.Ge, 03.65.Db, 98.80.Cq, 03.65.Pm

I. INTRODUCTION

Theory of topological defects is a natural framework for studying properties of physical systems. In cosmology, the origin of defects can be understood as a sequence of phase transitions in the early universe. These processes occur with critical temperatures which are related to the corresponding symmetry spontaneously breaking scales [1–3]. These phase transitions can give rise to topologically stable defects, for example, domain walls, strings and monopoles [4]. Topological defects are also found in condensed matter systems. In these systems, they appear as vortices in superconductors, domain wall in magnetic materials, dislocations of crystalline substances, among others. An important property that can be verified in topological defects is that they are described by a spacetime metric with a Riemann–Christoffel curvature tensor which is null everywhere except on the defects. Here, we look for a cosmic string, which is a linear topological defect with a conical singularity at the origin. The interest in this subject has contributed to the understanding and advancement of other physical phenomena occurring in the universe and also in the context of non-relativistic physics. For example, in the galaxy formation [5, 6], to study vortex solutions in non-abelian gauge theories with spontaneous symmetry breaking [7] and to study the gravitational analogue of the Aharonov–Bohm effect [8–12]. In recent developments, cosmic strings have

been considered to analyze solutions in de Sitter and anti-de Sitter spacetimes [13], to study the thermodynamic properties of a neutral particle in a magnetic cosmic string background by using an approach based on the partition function method [14], to compute the vacuum polarization energy of string configurations in models similar to the standard model of particle physics [15], to find the deflection angle in the weak limit approximation by a spinning cosmic string in the context of the Einstein–Cartan theory of gravity [16], to analyze numerically the behavior of the solutions corresponding to an Abelian string in the framework of the Starobinsky model [17], to study solutions of black holes [18], to investigate the average rate of change of energy for a static atom immersed in a thermal bath of electromagnetic radiation [19], to study Hawking radiation of massless and massive charged particles [20], to study the non-Abelian Higgs model coupled with gravity [21], in the quantum dynamics of scalar bosons [22], hydrodynamics [23], to study the non-relativistic motion of a quantum particle subjected to magnetic field [24], to investigate dynamical solutions in the context of super-critical tensions [25], Higgs condensate [26], to analyze the effects on spin current and Hall electric field [27, 28], to investigate the dynamics of the Dirac oscillator [29, 30], to study non-inertial effects on the ground state energy of a massive scalar field [31], Landau quantization [32] and to investigate the quantum vacuum interaction energy [33].

In the present work, we study the quantum dynamics of a spin-1/2 neutral particle in the presence of an electric field due to an infinitely long, infinitesimally thin line of charge along the z -axis of the cosmic string, with constant charge density on it. This model have been

* fmandrade@uepg.br

† cleversonfilgueiras@yahoo.com.br

‡ edilbertoo@gmail.com

studied in Ref. [34] in the non-relativistic regime and, for this particular case, only the scattering problem was considered. The present system is an adaptation of the usual Aharonov-Casher problem [35] (which is dual to the Aharonov-Bohm problem [36]), where now effects of localized curvature are included in the model. We reexamine this problem by using the von Neumann theory of self-adjoint extensions [37, 38]. We address the relativistic case and investigate some questions that were not considered in the previous studies, as for example, the existence of bound states. For this, we solve the scattering problem and derive the S matrix in order to obtain such bound states.

The plan of this work is the following. In Section II, we derive the Dirac-Pauli equation in the cosmic string spacetime without neglecting the term which depends explicitly on the spin. Arguments based on the theory of self-adjoint extension are given in order to make clear the reasons why we should consider the spin effects in the dynamics of the system. In Section III, we study the Dirac-Pauli Hamiltonian via the von Neumann theory of self-adjoint extension. We address the scattering scenario within the framework of Dirac-Pauli equation. Expressions for the phase shift, S-matrix, and bound states are derived. We also make an investigation on the helicity conservation problem in the present framework. A brief conclusion is outlined in Section IV.

II. THE RELATIVISTIC EQUATION OF MOTION

The model that we address here consists of a spin-1/2 neutral particle with mass M and magnetic moment μ , moving in an external electromagnetic field $F_{\mu\nu}$ in the cosmic string spacetime, described by the line element in cylindrical coordinates,

$$ds^2 = c^2 dt^2 - dr^2 - \alpha^2 r^2 d\varphi^2 - dz^2, \quad (1)$$

with $-\infty < (t, z) < \infty$, $r \geq 0$, $0 \leq \varphi \leq 2\pi$ and α is given in terms of the linear mass density \tilde{m} of the cosmic string by $\alpha = 1 - 4\tilde{m}/c^2$. This metric has a cone-like singularity at $r = 0$ [39]. In this system, the fermion particle is described by a four-component spinorial wave function Ψ obeying the generalized Dirac-Pauli equation in a non flat spacetime, which should include the spin connection in the differential operator. Moreover, in order to make the Dirac-Pauli equation valid in curved spacetime, we must rewrite the standard Dirac matrices, which are written in terms of the local coordinates in the Minkowski spacetime, in terms of global coordinates. This can be accomplished by using the inverse vierbeins $e_{\bar{a}}^\mu$ through the relation $\gamma^\mu = e_{\bar{a}}^\mu \gamma^{\bar{a}}$ ($\mu, \bar{a} = 0, 1, 2, 3$), with $\gamma^{\bar{a}} = (\gamma^{\bar{0}}, \gamma^{\bar{i}})$ being the standard gamma matrices. The equation of motion governing the dynamics of this system is the modified Dirac-Pauli equation in the curved

space

$$\left[i\hbar \gamma^\mu (\partial_\mu + \Gamma_\mu) - \frac{\mu}{2c} \sigma^{\mu\nu} F_{\mu\nu} - Mc \right] \Psi = 0, \quad (2)$$

with $\sigma^{\mu\nu} = i[\gamma^\mu, \gamma^\nu]/2$, $(F_{0i}, F_{ij}) = (E^i, \epsilon_{ijk} B^k)$, $(\sigma^{0j}, \sigma^{ij}) = (i\alpha^j, -\epsilon_{ijk} \Sigma^k)$, where E^i and B^k are the electric and magnetic field strengths and Σ^k is the spin operator. Here, we use the same vierbein of the Ref. [40], where the spinorial affine connection Γ_μ has been calculated in detail. Moreover, in this work, we are only interested on the planar dynamics of a spin-1/2 neutral particle under the action of a radial electric field. In this manner we require that $p_z = z = 0$ and $B^k = 0$ for $k = 1, 2, 3$. Furthermore, according to the tetrad postulated [41], the matrices $\gamma^{\bar{a}}$ can be any set of constant Dirac matrices in a such way that we are free to choose a representation for them. We choose to work in a representation in which the Dirac matrices are given in terms of the Pauli matrices, namely [42, 43]

$$\beta = \gamma^{\bar{0}} = \sigma^3, \quad \gamma^{\bar{1}} = i\sigma^2, \quad \gamma^{\bar{2}} = -is\sigma^1, \quad (3)$$

where $(\sigma^1, \sigma^2, \sigma^3)$ are the Pauli matrices and s is twice the spin value, with $s = +1$ for spin “up” and $s = -1$ for spin “down”. In this representation, the only non-vanishing component of the spinorial affine connection Γ_μ is found to be

$$\Gamma_\varphi = -i \frac{(1-\alpha)}{2} s \sigma^z. \quad (4)$$

For the field configuration, we consider the electric field due to a linear charge distribution, superposed to the cosmic string. The expression for this field is seen to be

$$E_r = \frac{2\lambda}{\alpha r}. \quad (5)$$

Therefore, the second order equation associated with Eq. (2) reads

$$\hat{H}\Phi = k^2\Phi, \quad (6)$$

with

$$\begin{aligned} \hat{H} = & -\nabla_\alpha^2 - \frac{(1-\alpha)s\sigma^z}{i\alpha^2 r^2} \partial_\varphi + \frac{(1-\alpha)^2}{4\alpha^2 r^2} + \frac{2\mu s}{\hbar c} \frac{E_r}{i\alpha r} \partial_\varphi \\ & + \frac{\mu}{\hbar c} (\partial_r E_r) \sigma^z - \frac{\mu}{\hbar c} \frac{(1-\alpha)E_r}{\alpha r} \sigma^z + \frac{\mu^2}{\hbar^2 c^2} E_r^2, \end{aligned} \quad (7)$$

where $\nabla_\alpha^2 = \partial_r^2 + (1/r)\partial_r + (1/\alpha^2 r^2)\partial_\varphi^2$ is the Laplace-Beltrami operator in the conical space and $k^2 = (\mathcal{E}^2 - M^2 c^4)/\hbar^2 c^2$. As the angular momentum $\hat{J} = -i\partial_\varphi + (s/2)\sigma^z$, commutes with the \hat{H} , it is possible to decompose the fermion field as

$$\Phi = \begin{pmatrix} \psi \\ \chi \end{pmatrix} = \begin{pmatrix} \sum_m f_m(r) e^{im\varphi} \\ \sum_m g_m(r) e^{i(m+s)\varphi} \end{pmatrix}, \quad (8)$$

where $m = 0, \pm 1, \pm 2, \pm 3, \dots$ is the angular momentum quantum number. In this manner, the radial equation for $f_m(r)$ is

$$hf_m(r) = k^2 f_m(r), \quad (9)$$

with

$$h = h_0 + \frac{\eta}{\alpha} \frac{\delta(r)}{r}, \quad (10)$$

and

$$h_0 = -\frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} + \frac{j^2}{r^2}, \quad (11)$$

where

$$j = \frac{m + s\eta}{\alpha} - \frac{s(1 - \alpha)}{2\alpha}, \quad (12)$$

is the effective angular momentum and

$$\eta = \frac{\phi}{\phi_0}. \quad (13)$$

Here, $\phi = 4\pi\lambda$ is the electric flux of the electric field and $\phi_0 = hc/\mu$ is the quantum of electric flux.

As far as we know, only the scattering problem for the Hamiltonian in Eq. (10) has been studied in Ref. [34]. However, there, the spin effect was not taken into account once the author imposed the regularity of the wave function at the origin. The inclusion of spin gives rise to the Dirac delta function potential, which comes from the interaction between the spin and the line of charge, and its inclusion has effects on the scattering phase shift, giving rise to an additional scattering phase shift [44]. Thus, the main aim of this work is to show that there are bound states due to the presence of the Dirac delta function. The approach adopted here is that of the self-adjoint extensions [38], which has been used to deal with singular Hamiltonians, for instance, in the study of spin 1/2 Aharonov-Bohm system and cosmic strings [45, 46], in the Aharonov-Bohm-Coulomb problem [47–50], and in the equivalence between the self-adjoint extension and normalization [51].

III. SCATTERING AND BOUND STATES ANALYSIS

In this section, we obtain the S-matrix and from its poles an expression for the bound states is obtained. Before we solve Eq. (9), let us first analyze the Hamiltonian h_0 .

In the von Neumann theory of self-adjoint extensions, a Hermitian operator \hat{O} ($\hat{O} = \hat{O}^\dagger$) defined in a dense subset of a Hilbert space has deficiency indices (n_+, n_-) , which are the sizes of the deficiency sub-spaces spanned by the solutions for

$$\hat{O}\chi_\pm = \pm i\chi_\pm. \quad (14)$$

When the dimension of the deficiency subspace are zero, the operator is self-adjoint and it has no additional self-adjoint extension. When the dimension of the deficiency spaces are not zero the operator is not self-adjoint. If $n_+ = n_- = n$ the operator admits a self-adjoint extension parametrized by a $n \times n$ unitary matrix. However, if the deficiency indices are not equal, the operator has no self-adjoint extensions. By standard results, it is well-known that the Hamiltonian h_0 has deficiency indices $(1, 1)$ and it is self-adjoint for $|j| \geq 1$, whereas for $|j| < 1$ it is not self-adjoint, and admits an one-parameter family of self-adjoint extensions [52]. Actually, h can be interpreted as a self-adjoint extension of h_0 [53]. All the self-adjoint extension of h_0 , $h_{0,\nu}$, are accomplished by requiring the boundary condition at the origin [37]

$$\nu f_{0,j} = f_{1,j}, \quad (15)$$

where $-\infty < \nu \leq \infty$ and $-1 < j < 1$. The boundary values are

$$f_{0,j} = \lim_{r \rightarrow 0^+} r^{|j|} f_m(r),$$

$$f_{1,j} = \lim_{r \rightarrow 0^+} \frac{1}{r^{|j|}} \left[f_m(r) - f_{0,j} \frac{1}{r^{|j|}} \right].$$

In Eq. (15) ν is the self-adjoint extension parameter. It turns out that $1/\nu$ represents the scattering length of $h_{0,\nu}$ [38]. For $\nu = \infty$ (the Friedrichs extension of h_0), one has the free Hamiltonian (without spin) with regular wave functions at the origin ($f_m(0) = 0$). This situation is equivalent to impose the Dirichlet boundary condition on the wave function. On the other hand, if $|\nu| < \infty$, $h_{0,\nu}$ describes a point interaction at the origin. In this latter case the boundary condition permits a $r^{-|j|}$ singularity in the wave functions at the origin [54].

Let us now discuss for which values of the angular momentum quantum number m , the operator h_0 is not self-adjoint. In fact, these values depending on the variables α and η . As discussed in [34], $0 < \alpha < 1$ represents a positive curvature and a planar deficit angle, corresponding to a conical spacetime. On the other hand, $\alpha > 1$ represents a negative curvature and an excess of planar angle, corresponding to an anti-conical spacetime. Finally, $\alpha = 1$ corresponds to a flat space. Then, we focus on a conical spacetime. For the electric flux η let us adopt the decomposition defined by [55]

$$\eta = N + \beta, \quad (16)$$

being N an integer and

$$0 \leq \beta < 1. \quad (17)$$

The inequality $|j| < 1$ then reads

$$\pi_-(\alpha, \beta) < m < \pi_+(\alpha, \beta), \quad (18)$$

with $\pi_\pm(\alpha, \beta) = \pm\alpha - [(2\beta + \alpha - 1)s/2 - sN]$. In Fig. 1 we plot the planes $\pi_\pm(\alpha, \beta)$ for $N = 0$ and $s = +1$.

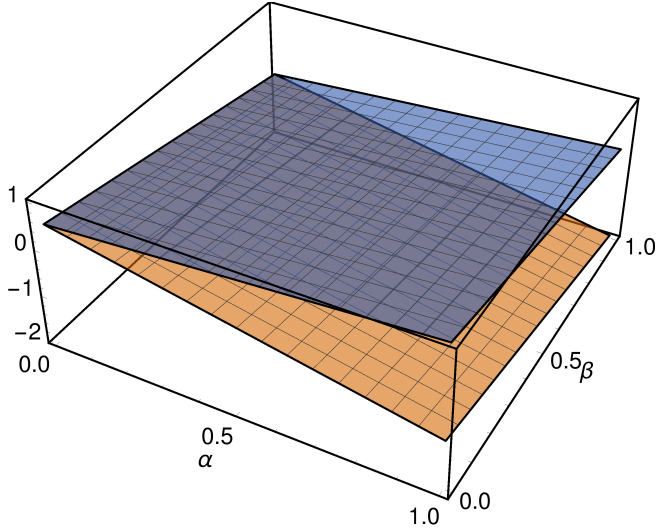


FIG. 1. In this figure we plot the planes $\pi_-(\alpha, \beta)$ (orange, bottom) and $\pi_+(\alpha, \beta)$ (blue, top) for $N = 0$ and $s = +1$. The region between the two planes is that in which the operator h_0 is not self-adjoint.

The region between these two planes is that in which the operator h_0 is not self-adjoint. In Fig. 2 we show cross sections of this region for some particular values of the deficit angle α . We can observe in Fig. 2(a) that, for $\alpha = 0.25$, only for $m = 0$ the operator h_0 is not self adjoint, whereas for $\alpha = 0.50$ (Fig. 2(b)) the operator h_0 is not self adjoint for $m = 0$ and $m = -1$, but not for both values of m at same time for the whole range of β values. Indeed, a necessary condition for the operator h_0 not being self-adjoint for the state with $m = -1$ is $\alpha > 1 - 2\beta/3$. In fact, this condition is also valid for $N \neq 0$ and, in this latter case, the m values for which the h_0 is not self-adjoint are shifted to the values sN and $sN - 1$. For $\alpha = 0.75$ (Fig. 2(c)) we can observe that there is a range the values of β in which, for both values of $m = 0$ and $m = -1$, the operator h_0 is not self-adjoint. And last but not least, $\alpha = 1.0$ (see Fig 2 (d)) is the only situation in which the operator h_0 is not self-adjoint for the both values of angular momentum quantum number for the whole range of β (the unique exception is $\beta = 0$).

Now, let us comeback to the solution of Eq. (9). As a matter of fact, it is the Bessel differential equation. Thus, the general solution for $r \neq 0$ is seen to be

$$f_m(r) = a_m J_{|j|}(kr) + b_m J_{-|j|}(kr), \quad (19)$$

where $J_\nu(z)$ is the Bessel function of fractional order. The coefficients a_m and b_m represent the contributions of the regular and irregular solutions at the origin, respectively. Thus making use of the boundary condition in Eq. (15) in the subspace $|j| < 1$, a relation between the coefficients is obtained, namely,

$$b_m = -\mu_\nu a_m, \quad (20)$$

where the term μ_ν is given by

$$\mu_\nu = \frac{k^{2|j|}\Gamma(1 - |j|)\sin(|j|\pi)}{4|j|\Gamma(1 + |j|)\nu + k^{2|j|}\Gamma(1 - |j|)\cos(|j|\pi)}, \quad (21)$$

where $\Gamma(z)$ is the gamma function. Therefore, in this subspace the solution reads

$$f_m(r) = a_m [J_{|j|}(kr) - \mu_\nu J_{-|j|}(kr)]. \quad (22)$$

The above equation shows that the self-adjoint extension parameter ν controls the contribution of the irregular solution $J_{-|j|}$ for the wave function. As a result, for $\nu = \infty$, we have $\mu_\infty = 0$, and there is no contribution of the irregular solution at the origin for the wave function. Consequently, the total wave function reads

$$\psi = \sum_{m=-\infty}^{\infty} a_m J_{|j|}(kr) e^{im\varphi}. \quad (23)$$

It is well-known that the coefficient a_m must be chosen in a such way that ψ represents a plane wave that is incident from the right. In this manner, we obtain the result

$$a_m = e^{-i|j|\pi/2}. \quad (24)$$

The scattering phase shift can be obtained from the asymptotic behavior of Eq. (23). This leads to

$$\delta_m = \frac{\pi}{2}(|m| - |j|). \quad (25)$$

This is the scattering phase shift of the Aharonov-Casher effect in the cosmic string background. It is worthwhile to note that, for $\alpha = 1$, it reduces to the phase shift for the usual Aharonov-Casher effect in flat space $\delta_m = \pi(|m| - |m + s\eta|)/2$ [35].

On the other hand, for $|\nu| < \infty$, the contribution of the irregular solution modifies the scattering phase shift to

$$\delta_m^\nu = \delta_m + \arctan(\mu_\nu). \quad (26)$$

Thus one obtains

$$S_m^\nu = e^{2i\delta_m^\nu} = e^{2i\delta_m} \left(\frac{1 + i\mu_\nu}{1 - i\mu_\nu} \right), \quad (27)$$

which is the expression for the S-matrix in terms of the phase shift. As a result, one observes that in this latter case there is an additional scattering for any value of the self-adjoint extension parameter ν . When $\nu = \infty$, we have the S-matrix for the Aharonov-Casher effect on the cosmic string background, as it should be.

The S-matrix or scattering matrix relates incoming and outgoing wave functions of a physical system undergoing a scattering process. Bound states are identified as the poles of the S-matrix in the upper half in the complex k plane. In this manner, the poles are determined at the zeros of the denominator in Eq. (27) with the replacement $k \rightarrow i\kappa$ with $\kappa = \sqrt{-(\mathcal{E}^2 - M^2 c^4)/\hbar^2 c^2}$. Therefore, for

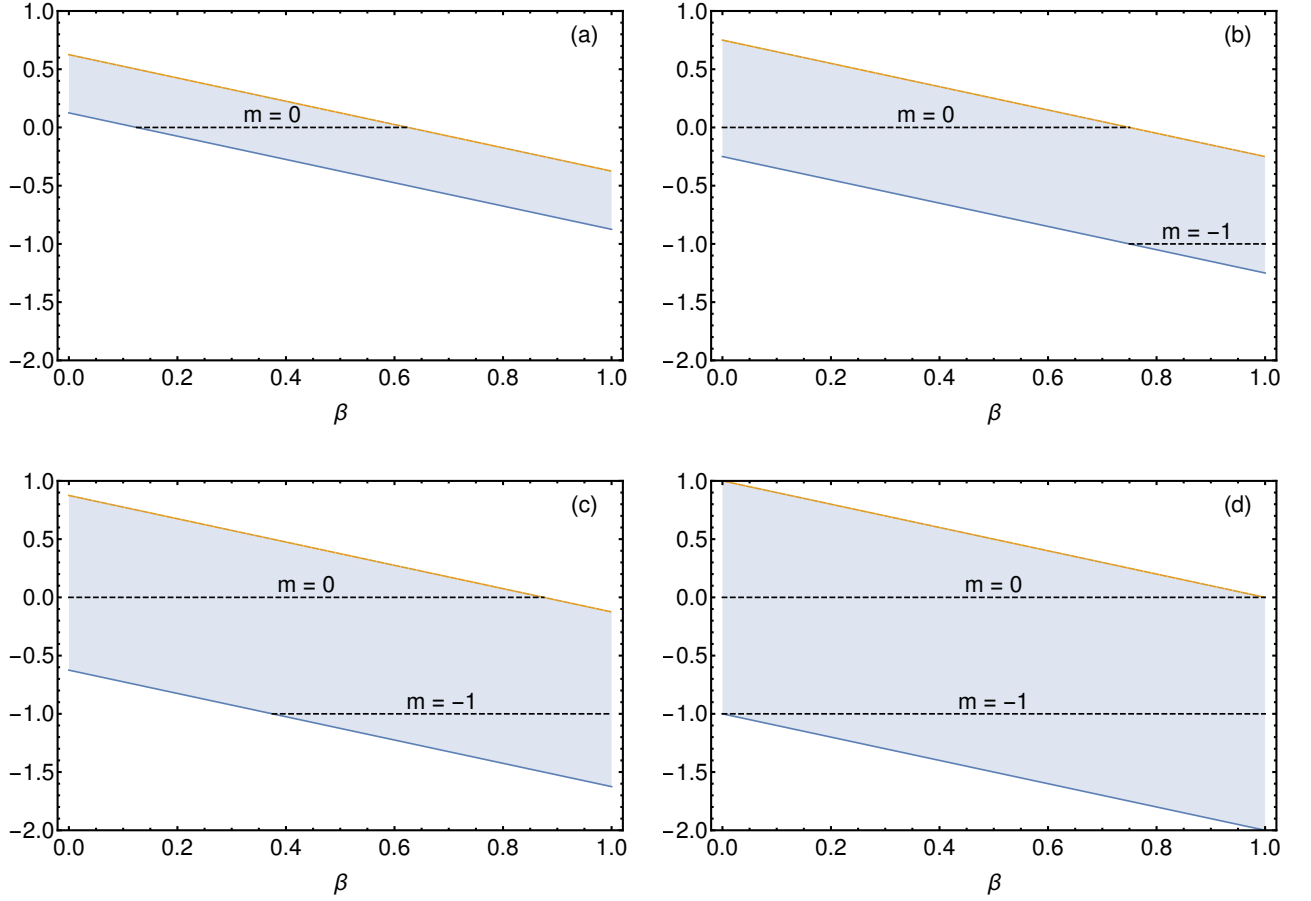


FIG. 2. Cross sections of Fig. 1 for different values of the deficit angle: (a) $\alpha = 0.25$, (b) $\alpha = 0.50$, (c) $\alpha = 0.75$ and (d) $\alpha = 1.00$. The shaded area schematically represents that area in which the operator h_0 is not self-adjoint. The dashed lines represent the values of angular momentum quantum number.

$\nu < 0$, one can determine that the present system has a bound state with energy given

$$\mathcal{E} = \pm \sqrt{M^2 c^4 - 4 \hbar^2 c^2 \left[-\nu \frac{\Gamma(1 + |j|)}{\Gamma(1 - |j|)} \right]^{1/|j|}}. \quad (28)$$

and the normalized radial bound state wave function is

$$\begin{aligned} \begin{pmatrix} f_m(r) \\ g_m(r) \end{pmatrix} &= \sqrt{\frac{2\alpha\kappa^2/\pi}{|j|\alpha \csc(|j|\pi) + |j'| \csc(|j'|\pi)}} \\ &\times \begin{pmatrix} K_{|j|}(\kappa r) \\ K_{|j'|}(\kappa r) \end{pmatrix}, \end{aligned} \quad (29)$$

where $j' = j + (s/\alpha)(2 - \alpha)$ and $K_\nu(z)$ is the modified Bessel function of the second kind. So, there are bound states when the self-adjoint extension parameter is negative. In the non-relativistic limit and for $\alpha = 1$, Eq. (28) coincides with the bound state energy found in Ref. [56] for the Aharonov-Casher effect in the flat space.

As a result, it is possible to express the S-matrix in terms of the bound state energy. The result is seem to

be

$$S_m^\nu = e^{2i\delta_m} \left[\frac{e^{2i\pi|j|} - (\kappa/k)^{2|j|}}{1 - (\kappa/k)^{2|j|}} \right]. \quad (30)$$

Once we have obtained the S-matrix, it is possible to write down the scattering amplitude $f(k, \varphi)$. The result is

$$\begin{aligned} f(k, \varphi) &= \frac{1}{\sqrt{2\pi i k}} \sum_{m=-\infty}^{\infty} (S_m^\nu - 1) e^{im\varphi} \\ &= \frac{1}{\sqrt{2\pi i k}} \left\{ \sum_{m \in \{|j| \geq 1\}} (e^{2i\delta_m} - 1) e^{im\varphi} \right. \\ &\quad \left. + \sum_{m \in \{|j| < 1\}} \left[e^{2i\delta_m} \left(\frac{1 + i\mu_\nu}{1 - i\mu_\nu} \right) - 1 \right] e^{im\varphi} \right\}. \end{aligned} \quad (31)$$

In scattering problems the length scale is set by $1/k$, thus the scattering amplitude $f(k, \varphi)$ would be a function of angle alone, multiplied by $1/k$ [57]. However, we observe that $f(k, \varphi)$ has a dependence on μ_ν , which in its turn has

explicit dependence on k (see Eq. (21)). This behavior is associated with the failure of helicity conservation. The helicity operator, defined by

$$\hat{h} = \boldsymbol{\Sigma} \cdot (-i\nabla_{\alpha} - e\mathbf{A}), \quad (32)$$

obeys the equation

$$\frac{d\hat{h}}{dt} = e\boldsymbol{\Sigma} \cdot \mathbf{E}, \quad (33)$$

whit $\boldsymbol{\Sigma}$ is the spin operator and in Eq. (32) \mathbf{A} is the potential vector, which is absent in the present problem. Therefore, due to the presence of electric field the helicity is not conserved.

IV. CONCLUSIONS

In this work, we reexamined the relativistic quantum dynamics of a spin-1/2 neutral particle in the cosmic string spacetime. This problem has been studied in Ref. [34] in the non-relativistic scenario. However only the scattering solutions were studied and without taking into account the possibility of bound states. Here, we have showed that the inclusion of electron spin, which gives rise to a point interaction, changes the scattering phase shift and consequently the S-matrix. The results were obtained by imposing the boundary condition in Eq. (15),

which comes from the von Neumann theory of the self-adjoint extensions. Our results are dependent on the self-adjoint extension parameter ν . For the special value of $\nu = \infty$ we recover the results of Ref. [34]. Our expression for the scattering amplitude has an energy dependency. So, the helicity is not conserved in the scattering process. Last but not least, examining the poles of the S-matrix, an expression for the bound state energy was determined. The presence of bound states has not been discussed before.

CONFLICT OF INTEREST

The authors declare that there is no conflict of interests regarding the publication of this paper.

ACKNOWLEDGEMENTS

FMA thanks Simone Severini and Sougato Bose by their hospitality at University College London. This work was partially supported by the CNPq, Brazil, Grants Nos. 482015/2013-6 (Universal), 476267/2013-7 (Universal), 460404/2014-8 (Universal), 306068/2013-3 (PQ), 311699/2014-6 (PQ), FAPEMA, Brazil, Grants No. 01852/14 (PRONEM) and FAPEMIG.

-
- [1] L. Dolan and R. Jackiw, *Phys. Rev. D* **9**, 3320 (1974).
 - [2] S. Weinberg, *Phys. Rev. D* **9**, 3357 (1974).
 - [3] A. D. Linde, *Rep. Prog. Phys.* **42**, 389 (1979).
 - [4] T. W. B. Kibble, *J. Phys. A: Math. Gen.* **9**, 1387 (1976).
 - [5] J. Silk and A. Vilenkin, *Phys. Rev. Lett.* **53**, 1700 (1984).
 - [6] N. Turok and R. H. Brandenberger, *Phys. Rev. D* **33**, 2175 (1986).
 - [7] M. Manías, C. Naón, F. Schaposnik, and M. Trobo, *Phys. Lett. B* **171**, 199 (1986).
 - [8] J. S. Dowker, *Il Nuovo Cimento B Series 10* **52**, 129 (1967).
 - [9] L. H. Ford and A. Vilenkin, *J. Phys. A: Math. Gen.* **14**, 2353 (1981).
 - [10] J. Stachel, *Phys. Rev. D* **26**, 1281 (1982).
 - [11] V. B. Bezerra, *Phys. Rev. D* **35**, 2031 (1987).
 - [12] A. Aliev and D. Gal'tsov, *Ann. Phys.* **193**, 142 (1989).
 - [13] A. de Pádua Santos and E. R. Bezerra de Mello, *Phys. Rev. D* **94**, 063524 (2016).
 - [14] H. Hassanabadi and M. Hosseinpour, *The European Physical Journal C* **76**, 553 (2016).
 - [15] H. Weigel, M. Quandt, and N. Graham, *Phys. Rev. D* **94**, 045015 (2016).
 - [16] K. Jusufi, *The European Physical Journal C* **76**, 332 (2016).
 - [17] J. P. M. Graa, *Classical and Quantum Gravity* **33**, 055004 (2016).
 - [18] H. Vieira, V. Bezerra, and G. Silva, *Ann. Phys.* **362**, 576 (2015).
 - [19] H. Cai, H. Yu, and W. Zhou, *Phys. Rev. D* **92**, 084062 (2015).
 - [20] K. Jusufi, *Gen. Relativ. Gravitation* **47**, 1 (2015).
 - [21] A. de Pádua Santos and E. R. Bezerra de Mello, *Class. Quantum Grav.* **32**, 155001 (2015).
 - [22] L. B. Castro, *Eur. Phys. J. C* **75**, 287 (2015).
 - [23] A. Beresnyak, *The Astrophysical Journal* **804**, 121 (2015).
 - [24] H. Hassanabadi, A. Afshardoost, and S. Zarrinkamar, *Ann. Phys.* **356**, 346 (2015).
 - [25] F. Niedermann and R. Schneider, *Phys. Rev. D* **91**, 064010 (2015).
 - [26] H. F. S. Mota and M. Hindmarsh, *Phys. Rev. D* **91**, 043001 (2015).
 - [27] J.-h. Wang, K. Ma, and K. Li, *Phys. Rev. A* **87**, 032107 (2013).
 - [28] D. Chowdhury and B. Basu, *Phys. Rev. D* **90**, 125014 (2014).
 - [29] J. Carvalho, C. Furtado, and F. Moraes, *Phys. Rev. A* **84**, 032109 (2011).
 - [30] F. M. Andrade and E. O. Silva, *Eur. Phys. J. C* **74**, 3187 (2014), 1403.4113.
 - [31] H. F. Mota and K. Bakke, *Phys. Rev. D* **89**, 027702 (2014).
 - [32] C. Muniz, V. Bezerra, and M. Cunha, *Ann. Phys.* **350**, 105 (2014).
 - [33] J. M. M. Muñoz Castañeda and M. Bordag, *Phys. Rev. D* **89**, 065034 (2014).

- [34] E. R. Bezerra de Mello, *J. High Energy Phys.* **2004**, 016 (2004).
- [35] Y. Aharonov and A. Casher, *Phys. Rev. Lett.* **53**, 319 (1984).
- [36] Y. Aharonov and D. Bohm, *Phys. Rev.* **115**, 485 (1959).
- [37] W. Bulla and F. Gesztesy, *J. Math. Phys.* **26**, 2520 (1985).
- [38] S. Albeverio, F. Gesztesy, R. Hoegh-Krohn, and H. Holden, *Solvable Models in Quantum Mechanics*, 2nd ed. (AMS Chelsea Publishing, Providence, RI, 2004).
- [39] D. D. Sokolov and A. A. Starobinski, *Sov. Phys. Dokl.* **22**, 312 (1977).
- [40] K. Bakke, J. R. Nascimento, and C. Furtado, *Phys. Rev. D* **78**, 064012 (2008).
- [41] I. Lawrie, *A Unified Grand Tour of Theoretical Physics, Third Edition* (Taylor & Francis, 2012).
- [42] R. H. Brandenberger, A.-C. Davis, and A. M. Matheson, *Nucl. Phys. B* **307**, 909 (1988).
- [43] M. Alford, J. March-Russell, and F. Wilczek, *Nucl. Phys. B* **328**, 140 (1989).
- [44] C. R. Hagen, *Phys. Rev. Lett.* **64**, 503 (1990).
- [45] V. Khalilov, *Eur. Phys. J. C* **74**, 1 (2014).
- [46] P. de Sousa Gerbert, *Phys. Rev. D* **40**, 1346 (1989).
- [47] V. Khalilov, *Eur. Phys. J. C* **73**, 1 (2013).
- [48] V. Khalilov and I. Mamsurov, *Theor. Math. Phys.* **161**, 1503 (2009).
- [49] D. K. Park and J. G. Oh, *Phys. Rev. D* **50**, 7715 (1994).
- [50] D. K. Park, *J. Math. Phys.* **36**, 5453 (1995).
- [51] R. Jackiw, *Diverse topics in theoretical and mathematical physics*, Advanced Series in Mathematical Physics (World Scientific, Singapore, 1995).
- [52] M. Reed and B. Simon, *Methods of Modern Mathematical Physics. II. Fourier Analysis, Self-Adjointness*. (Academic Press, New York - London, 1975).
- [53] F. Gesztesy, S. Albeverio, R. Hoegh-Krohn, and H. Holden, *Journal für die reine und angewandte Mathematik (Crelles Journal)*, *J. Reine Angew. Math.* **380**, 87 (1987).
- [54] C. R. Hagen, *Phys. Rev. A* **77**, 036101 (2008).
- [55] C. R. Hagen, *Phys. Rev. D* **41**, 2015 (1990).
- [56] E. O. Silva, F. M. Andrade, C. Filgueiras, and H. Belich, *Eur. Phys. J. C* **73**, 2402 (2013).
- [57] A. S. Goldhaber, *Phys. Rev. D* **16**, 1815 (1977).